

On the g -Circulant Matrix involving the Generalized k -Horadam Numbers

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Abstract

In this study, we present a new generalization of circulant matrices for the generalized k -Horadam numbers, by considering the g -circulant matrix $C_{n,g}(H) = g - \text{circ}(H_{k,1}, H_{k,2}, \dots, H_{k,n})$. Also, we calculate the spectral norm, determinant and inverse of $C_{n,g}(H)$ in such matrices having the elements of all second order sequences.

Keywords: Determinant, g -circulant matrix, Generalized k -Horadam number, Inverse.

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1 Introduction and Preliminaries

Many generalizations of the Fibonacci sequence have been introduced and studied [8, 12, 17, 23, 25]. Here we use the generalized k -Horadam numbers as follows.

Let k be any positive real number and $f(k)$, $g(k)$ are scalar-value polynomials and $f^2(k) + 4g(k) > 0$. For $n \geq 0$, the generalized k -Horadam sequence $\{H_{k,n}\}_{n \in \mathbb{N}}$ is defined by

$$H_{k,n+2} = f(k)H_{k,n+1} + g(k)H_{k,n}, H_{k,0} = a, H_{k,1} = b. \quad (1.1)$$

where $a, b \in \mathbb{R}$ [25]. Obviously, if we choose suitable values on $f(k), g(k), a$ and b in (1.1) then this sequence reduces to the special all second order sequences in the literature. For example, by taking $f(k) = g(k) = 1$, $a = 0$ and $b = 1$, then it is obtained the well known Fibonacci sequence.

Let r_1 and r_2 be the roots of the characteristic equation $x^2 - f(k)x - g(k) = 0$ of (1.1). Then the Binet formula of this sequence $\{H_{k,n}\}_{n \in \mathbb{N}}$ have the form

$$H_{k,n} = \frac{Xr_1^n - Yr_2^n}{r_1 - r_2}, \quad (1.2)$$

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where $X = b - ar_2$ and $Y = b - ar_1$. Also, the summation of this sequence is given by

$$\sum_{i=1}^n H_{k,i} = \frac{H_{k,n+1} + g(k)H_{k,n} - H_{k,1} - g(k)H_{k,0}}{f(k) + g(k) - 1}, \quad (1.3)$$

where $f(k) + g(k) - 1 \neq 0$.

The g -circulant matrices have been one of the most important and active research field of applied mathematic and computation mathematic increasingly. There are lots of examples from statistical and information theory illustrate applications of the g -circulant matrices, which emphasize how the asymptotic eigenvalue distribution theorem allows one to evaluate results for processes (see [7, 9, 13] and therein). In the last years, there have been several papers on circulant matrices [1]-[7],[9, 10],[14, 15, 16],[18]-[22],[26]-[30]. For instance, Alptekin, Mansour and Tuglu, [1], obtained the spectral norm and eigenvalues of circulant matrices with Horadam's numbers. Also, they defined the semicirculant matrix with these numbers and give Euclidean norm of this matrix. The authors in [15] defined g -circulant matrices with k -Fibonacci and k -Lucas numbers and computed the determinant and the inverse of these matrices. In [16], it was studied the norms, eigenvalues and determinants of some matrices related to different numbers. In [19], authors defined the $n \times n$ circulant matrices $A = [a_{ij}]$ and $B = [b_{ij}]$, where $a_{ij} \equiv F_{(\text{mod}(j-i,n))}$ and $b_{ij} \equiv L_{(\text{mod}(j-i,n))}$. Also, the inverses of matrices A and B were derived. In addition, Solak [20] defined the $n \times n$ circulant matrices $A = [a_{ij}]$ and $B = [b_{ij}]$, where $a_{ij} \equiv F_{(\text{mod}(j-i,n))}$ and $b_{ij} \equiv L_{(\text{mod}(j-i,n))}$. He investigated the upper and lower bounds of the matrices A and B . Additionally, Yazlik and Taskara [26, 27] defined circulant matrix $C_n(H)$ whose entries are the generalized k -Horadam numbers and computed the spectral norm, eigenvalues, determinant and the inverse of this matrix. That is, authors gave the determinant and inverse of matrix $C_n(H)$ as follows:

$$\det C_n(H) = H_{k,1}N^{n-1} + H_{k,1}M^{n-2} \sum_{i=1}^{n-1} \left(-\frac{H_{k,2}H_{k,i+1}}{H_{k,1}} + H_{k,i+2} \right) \left(\frac{N}{M} \right)^{i-1}, \quad (1.4)$$

and for $n > 2$,

$$\begin{aligned} C_n^{-1}(H) = & \text{circ} \left(\frac{1 + f(k)S_n^{(n-2)} + g(k)S_n^{(n-3)}g(k)S_n^{(n-2)} - \frac{H_{k,2}}{H_{k,1}}}{h_n}, \right. \\ & -\frac{S_n^{(1)}}{h_n}, -\frac{S_n^{(2)} - f(k)S_n^{(1)}}{h_n}, -\frac{S_n^{(3)} - f(k)S_n^{(2)} - g(k)S_n^{(1)}}{h_n} \\ & \left. , \dots, -\frac{S_n^{(n-2)} - f(k)S_n^{(n-3)} - g(k)S_n^{(n-4)}}{h_n} \right), \end{aligned} \quad (1.5)$$

where

$$S_n^{(j)} = \sum_{i=1}^j \frac{\left(H_{k,j+3-i} - \frac{H_{k,2}H_{k,j+2-i}}{H_{k,1}} \right) M^{i-1}}{N^i}, \quad (j = 1, 2, \dots, n-2),$$

$$\begin{aligned}
h_n &= -\frac{H_{k,2}H_{k,n}}{H_{k,1}} + H_{k,1} + \sum_{i=1}^{n-1} \left(-\frac{H_{k,2}H_{k,i+1}}{H_{k,1}} + H_{k,i+2} \right) \left(\frac{M}{N} \right)^{n-(i+1)}, \\
M &= g(k)(H_{k,n} - H_{k,0}) \text{ and } N = H_{k,1} - H_{k,n+1}.
\end{aligned}$$

Now we give some preliminaries related our study. A , g -circulant matrix is an $n \times n$ complex matrix with the following form

$$A = \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-g} & a_{n-g+1} & \cdots & a_{n-g-1} \\ a_{n-2g} & a_{n-2g+1} & \cdots & a_{n-2g-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_g & a_{g+1} & \cdots & a_{g-1} \end{pmatrix},$$

where g is nonnegative integer and each of the subscripts is understood to be reduced modulo n . The first row of A is $(a_0, a_1, \dots, a_{n-1})$ and its $(j+1)$ -th row is obtained by giving j -th row a right circular shift by g positions. Note that, $g = 1$ or $g = n+1$ yields the classical circulant matrix [21].

From [13], we further remind that, for a matrix $A = [a_{i,j}] \in M_{m,n}(\mathbb{C})$, the spectral norm of A is given by

$$\|A\|_2 = \sqrt{\max_{1 \leq i \leq n} \lambda_i(A^*A)}$$

where A^* is the conjugate transpose of matrix A .

Lemma 1.1 [13] *Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. If A is a normal matrix, then*

$$\|A\|_2 = \max_{1 \leq i \leq n} |\lambda_i|.$$

Lemma 1.2 [21] *An $n \times n$ matrix Q_g is unitary if and only if*

$$(n, g) = 1,$$

where Q_g is a g -circulant matrix with the first row $e^* = (1, 0, \dots, 0)$.

Lemma 1.3 [21] *A is g -circulant matrix with the first row $(a_0, a_1, \dots, a_{n-1})$ if and only if*

$$A = Q_g C,$$

where C is circulant matrix, that is, $C = \text{circ}(a_0, a_1, \dots, a_{n-1})$.

In the light of all these above material (depicted as separate paragraphs), the main goal of this paper is to investigate the properties of g -circulant matrix with k -Horadam numbers. To do that we consider g -circulant matrix $C_{n,g}(H) = g\text{-circ}(H_{k,1}, H_{k,2}, \dots, H_{k,n})$, where $H_{k,n}$ is the generalized k -Horadam numbers. Firstly, we obtain the values of the spectral norm and determinant of this matrix can be expressed with only the generalized k -Horadam numbers. Also we formulate the inverse of g -circulant matrix $C_{n,g}(H)$. In fact, the results in here are the most general statements to obtain the spectral norms, determinants and inverses in such matrices having the elements of all second order sequences.

2 Main Results

Definition 2.1 An $(n \times n)$ g -circulant matrix with generalized k -Horadam numbers entries is defined by

$$C_{n,g}(H) = \begin{pmatrix} H_{k,1} & H_{k,2} & H_{k,3} & \dots & H_{k,n} \\ H_{k,n-g+1} & H_{k,n-g+2} & H_{k,n-g+3} & \dots & H_{k,n-g} \\ H_{k,n-2g+1} & H_{k,n-2g+2} & H_{k,n-2g+3} & \dots & H_{k,n-2g} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{k,g+1} & H_{k,g+2} & H_{k,g+3} & \dots & H_{k,g} \end{pmatrix}, \quad (2.1)$$

where g is nonnegative integer.

The following theorem gives us the values of the determinant of this matrix can be expressed by utilizing the generalized k -Horadam numbers.

Theorem 2.1 Let $C_{n,g}(H) = g\text{-circ}(H_{k,1}, H_{k,2}, \dots, H_{k,n})$ be circulant matrix as in (2.1). Then we have

$$\|C_{n,g}(H)\|_2 = \frac{H_{k,n+1} + g(k)H_{k,n} - H_{k,1} - g(k)H_{k,0}}{f(k) + g(k) - 1},$$

where $f(k) + g(k) - 1 \neq 0$.

Proof. We express that g -circulant matrix is normal and irreducible (see [30]). So, the spectral norm of $C_{n,g}(H)$ is given by the spectral radius of $C_{n,g}(H)$. Also since $C_{n,g}(H)$ is irreducible and entrywise nonnegative, its spectral radius is the same as its Perron value. Let u denote an all ones vector of order n . Then $C_{n,g}(H)u = (\sum_{i=1}^n H_{k,i})u$. As $\sum_{i=1}^n H_{k,i}$ is an eigenvalue of $C_{n,g}(H)$ associated with a positive eigenvector, it is necessarily the Perron value of $C_{n,g}(H)$. Hence, from the Equation (1.3), we conclude that

$$\|C_{n,g}(H)\|_2 = \frac{H_{k,n+1} + g(k)H_{k,n} - H_{k,1} - g(k)H_{k,0}}{f(k) + g(k) - 1}.$$

■

Corollary 2.1 In Theorem 2.1, for special choices of $a, b, f(k)$ and $g(k)$, the following result can be obtained for well-known number sequences in literature:

- If $f(k) = 1$, $g(k) = 1$, $a = 0$ and $b = 1$, for the Fibonacci sequence in [30], we obtain $\|C_{n,g}(F)\|_2 = F_{n+2} - 1$,
- If $f(k) = 1$, $g(k) = 1$, $a = 2$ and $b = 1$, for the Lucas sequence in [30], we obtain $\|C_{n,g}(L)\|_2 = L_{n+2} - 3$,
- If $f(k) = 2$, $g(k) = 1$, $a = 0$ and $b = 1$, for the Pell sequence, we obtain $\|C_{n,g}(P)\|_2 = \frac{P_{n+1} + P_n - 1}{2}$,

- If $f(k) = 1$, $g(k) = 2$, $a = 0$ and $b = 1$, for the Jacobsthal sequence, we obtain $\|C_{n,g}(J)\|_2 = \frac{J_{n+2} - 1}{2}$,
- Finally, we should note that choosing suitable values on $f(k)$, $g(k)$, a and b in Theorem 2.1, it is actually obtained the spectral norms of g -circulant matrix for the others second order sequences such as k -Fibonacci, k -Lucas, Pell-Lucas, Jacobsthal-Lucas, Horadam, etc.

Theorem 2.2 Let $C_{n,g}(H) = g\text{-circ}(H_{k,1}, H_{k,2}, \dots, H_{k,n})$ be circulant matrix as in (2.1). Then we have

$$\det C_{n,g}(H) = \det Q_g \cdot \left[H_{k,1} N^{n-1} + H_{k,1} M^{n-2} \sum_{i=1}^{n-1} \left(-\frac{H_{k,2} H_{k,i+1}}{H_{k,1}} + H_{k,i+2} \right) \left(\frac{N}{M} \right)^{i-1} \right],$$

where $M = g(k)(H_{k,n} - H_{k,0})$, $N = H_{k,1} - H_{k,n+1}$ and $(n, g) = 1$.

Proof. By using Lemma 1.2 ve 1.3, we can write

$$C_{n,g}(H) = Q_g C_n(H),$$

where $(n, g) = 1$, Q_g is a g -circulant matrix and $C_n(H)$ is a circulant matrix with generalized k -Horadam number. From properties of determinant function and Equation (1.4), the proof is complete. ■

Corollary 2.2 In Theorem 2.2, for special choices of $a, b, f(k)$ and $g(k)$, the following result can be obtained for well-known number sequences in literature:

- If $f(k) = 1$, $g(k) = 1$, $a = 0$ and $b = 1$, for the Fibonacci sequence, we obtain $\det C_{n,g}(F) = \det Q_g \cdot \left[(1 - F_{n+1})^{n-1} + F_n^{n-2} \sum_{i=1}^{n-1} F_i \left(\frac{1 - F_{n+1}}{F_n} \right)^{i-1} \right]$
- If $f(k) = 1$, $g(k) = 1$, $a = 2$ and $b = 1$, for the Lucas sequence, we obtain $\det C_{n,g}(L) = \det Q_g \cdot \left[(1 - L_{n+1})^{n-1} + (L_n - 2)^{n-2} \sum_{i=1}^{n-1} (L_{i+2} - 3L_{i+1}) \left(\frac{1 - L_{n+1}}{L_n - 2} \right)^{i-1} \right]$
- If $f(k) = 2$, $g(k) = 1$, $a = 0$ and $b = 1$, for the Pell sequence, we obtain $\det C_{n,g}(P) = \det Q_g \cdot \left[(1 - P_{n+1})^{n-1} + P_n^{n-2} \sum_{i=1}^{n-1} P_i \left(\frac{1 - P_{n+1}}{P_n} \right)^{i-1} \right]$
- If $f(k) = 1$, $g(k) = 2$, $a = 0$ and $b = 1$, for the Jacobsthal sequence, we obtain $\det C_{n,g}(J) = \det Q_g \cdot \left[(1 - J_{n+1})^{n-1} + 2^{n-1} J_n^{n-2} \sum_{i=1}^{n-1} J_i \left(\frac{1 - J_{n+1}}{2J_n} \right)^{i-1} \right]$
- Finally, we should note that choosing suitable values on $f(k)$, $g(k)$, a and b in Theorem 2.2, it is actually obtained the determinant of g -circulant matrix for the others second order sequences such as k -Fibonacci, k -Lucas, Pell-Lucas, Jacobsthal-Lucas, Horadam, etc.

Proposition 2.1 Let $C_{n,g}(H) = g\text{-circ}(H_{k,1}, H_{k,2}, \dots, H_{k,n})$ be g -circulant matrix as in (2.1). For $n > 2$, $C_{n,g}(H)$ is an invertible matrix.

Proof. By using Lemma 1.2 ve 1.3, we can write $C_{n,g}(H) = Q_g C_n(H)$, where $(n, g) = 1$, Q_g is a g -circulant matrix and $C_n(H)$ is a circulant matrix with generalized k -Horadam number. From the Equation (1.5), $C_n(H)$ is invertible for $n > 2$. Hence, $C_{n,g}(H)$ is an invertible matrix, since $C_n(H)$ and Q_g are invertible. ■

Theorem 2.3 Let $C_{n,g}(H) = g\text{-circ}(H_{k,1}, H_{k,2}, \dots, H_{k,n})$ be g -circulant matrix as in (2.1), for $n > 2$, then, we have

$$C_{n,g}^{-1}(H) = [\text{circ}(\frac{1 + f(k)S_n^{(n-2)} + g(k)S_n^{(n-3)}}{h_n}, \frac{g(k)S_n^{(n-2)} - \frac{H_{k,2}}{H_{k,1}}}{h_n}, -\frac{S_n^{(1)}}{h_n}, \\ -\frac{S_n^{(2)} - f(k)S_n^{(1)}}{h_n}, -\frac{S_n^{(3)} - f(k)S_n^{(2)} - g(k)S_n^{(1)}}{h_n}, \dots, \\ -\frac{S_n^{(n-2)} - f(k)S_n^{(n-3)} - g(k)S_n^{(n-4)}}{h_n})] \cdot Q_g^T,$$

$$\text{where } S_n^{(j)} = \sum_{i=1}^j \frac{(H_{k,j+3-i} - \frac{H_{k,2}H_{k,j+2-i}}{H_{k,1}})M^{i-1}}{N^i} \quad (j = 1, 2, \dots, n-2), \quad h_n = -\frac{H_{k,2}H_{k,n}}{H_{k,1}} + \\ H_{k,1} + \sum_{i=1}^{n-1} \left(-\frac{H_{k,2}H_{k,i+1}}{H_{k,1}} + H_{k,i+2} \right) \left(\frac{M}{N} \right)^{n-(i+1)}, \quad M = g(k)(H_{k,n} - H_{k,0}) \text{ and } N = \\ H_{k,1} - H_{k,n+1}.$$

Proof. The proofs of theorem can be done similarly by considering Proposition 2.1. ■

Corollary 2.3 In Theorem 2.3, for special choices of a , b , $f(k)$ and $g(k)$, the following result can be obtained for well-known number sequences in literature:

- If $f(k) = 1$, $g(k) = 1$, $a = 0$ and $b = 1$, for the classic Fibonacci sequence, we obtain $C_{n,g}^{-1}(F) = \left[\frac{1}{f_n} \text{circ}(1 + \sum_{i=1}^{n-2} \frac{F_{n-i}F_{n-1}^{i-1}}{(F_1 - F_{n+1})^i}, -1 + \sum_{i=1}^{n-2} \frac{F_{n-1-i}F_{n-1}^{i-1}}{(F_1 - F_{n+1})^i}, -\frac{1}{F_1 - F_{n+1}}, \right. \\ \left. -\frac{F_n}{(F_1 - F_{n+1})^2}, -\frac{F_n^2}{(F_1 - F_{n+1})^3}, \dots, -\frac{F_n^{n-3}}{(F_1 - F_{n+1})^{n-2}}) \right] \cdot Q_g^T,$ where $f_n = F_1 - F_n + \sum_{i=1}^{n-2} F_i \left(\frac{F_n}{F_1 - F_{n+1}} \right)^{n-(i+1)}$.

- If $f(k) = 1$, $g(k) = 1$, $a = 2$ and $b = 1$, for the classic Lucas sequence, we obtain $C_{n,g}^{-1}(L) = \left[\frac{1}{l_n} \text{circ}(1 + \sum_{i=1}^{n-2} \frac{(L_{n+2-i} - 3L_{n+1-i})(L_n - 2)^{i-1}}{(L_1 - L_{n+1})^i}, \right. \\ \left. -3 + \sum_{i=1}^{n-2} \frac{(L_{n+1-i} - 3L_{n-i})(L_n - 2)^{i-1}}{(L_1 - L_{n+1})^i}, \frac{5}{L_1 - L_{n+1}}, \right. \\ \left. \frac{5(L_n - 2)}{(L_1 - L_{n+1})^2}, \frac{5(L_n - 2)^2}{(L_1 - L_{n+1})^3}, \dots, \frac{5(L_n - 2)^{n-3}}{(L_1 - L_{n+1})^{n-2}}) \right] \cdot Q_g^T,$ where $l_n = L_1 - 3L_n + \sum_{i=1}^{n-2} (L_{i+2} - 3L_{i+1}) \left(\frac{L_n - 2}{L_1 - L_{n+1}} \right)^{n-(i+1)}$.

- If $f(k) = 2$, $g(k) = 1$, $a = 0$ and $b = 1$, for the classic Pell sequence, we obtain

$$C_{n,g}^{-1}(P) = \begin{bmatrix} \frac{1}{p_n} \text{circ}(1 + \sum_{i=1}^{n-2} \frac{P_{n-i} P_n^{i-1}}{(P_1 - P_{n+1})^i}, -2 + \sum_{i=1}^{n-2} \frac{P_{n-1-i} P_n^{i-1}}{(P_1 - P_{n+1})^i}, \\ -\frac{1}{P_1 - P_{n+1}}, -\frac{P_n}{(P_1 - P_{n+1})^2}, -\frac{P_n^2}{(P_1 - P_{n+1})^3}, \dots, -\frac{P_n^{n-3}}{(P_1 - P_{n+1})^{n-2}} \end{bmatrix} \cdot Q_g^T,$$

where $p_n = P_1 - 2P_n + \sum_{i=1}^{n-2} P_i \left(\frac{P_n}{P_1 - P_{n+1}} \right)^{n-(i+1)}$.

- If $f(k) = 1$, $g(k) = 2$, $a = 0$ and $b = 1$, for the classic Jacobsthal sequence, we

$$\text{obtain } C_{n,g}^{-1}(J) = \begin{bmatrix} \frac{1}{s_n} \text{circ}(1 + 2 \sum_{i=1}^{n-2} \frac{J_{n-i}(2J_n)^{i-1}}{(J_1 - J_{n+1})^i}, -1 + 4 \sum_{i=1}^{n-2} \frac{J_{n-1-i}(2J_n)^{i-1}}{(J_1 - J_{n+1})^i}, \\ -\frac{2}{J_1 - J_{n+1}}, -\frac{2^2 J_n}{(J_1 - J_{n+1})^2}, -\frac{2^3 J_n^2}{(J_1 - J_{n+1})^3}, \dots, -\frac{2^{n-2} J_n^{n-3}}{(J_1 - J_{n+1})^{n-2}} \end{bmatrix}.$$

Q_g^T , where $s_n = J_1 - J_n + 2 \sum_{i=1}^{n-2} J_i \left(\frac{2J_n}{J_1 - J_{n+1}} \right)^{n-(i+1)}$.

- If $f(k) = p$ and $g(k) = q$, for the classic Horadam sequence, we obtain $C_{n,g}^{-1}(W) =$

$$\begin{bmatrix} \frac{1}{z_n} \text{circ}(1 + \sum_{i=1}^{n-2} \left(w_{n+2-i} - \frac{w_2}{w_1} w_{n+1-i} \right) \frac{A^{i-1}}{B^i}, \\ \frac{-w_2}{w_1} + q \sum_{i=1}^{n-2} \frac{\left(w_{n+1-i} - \frac{w_2 w_{n-i}}{w_1} \right) A^{i-1}}{B^i}, -\frac{w_3 - \frac{w_2^2}{B}}{B}, \\ -\frac{\left(w_3 - \frac{w_2^2}{w_1} \right) A}{B^2}, -\frac{\left(w_3 - \frac{w_2^2}{w_1} \right) A^2}{B^3}, \dots, -\frac{\left(w_3 - \frac{w_2^2}{w_1} \right) A^{n-3}}{B^{n-2}} \end{bmatrix} \cdot Q_g^T,$$

where $z_n = -\frac{w_2 w_n}{w_1} + w_1 + \sum_{i=1}^{n-2} \left(-\frac{w_2 w_{i+1}}{w_1} + w_{i+2} \right) \left(\frac{A}{B} \right)^{n-(i+1}$, $A = q(w_n - w_0)$ and $B = w_1 - w_{n+1}$.

- Finally, we should note that choosing suitable values on $f(k)$, $g(k)$, a and b in Theorem 2.3, it is actually obtained inverse of g -circulant matrix for the others second order sequences such as k -Fibonacci, k -Lucas, Pell-Lucas, Jacobsthal-Lucas, Horadam sequences.

Conclusion 2.1 In this paper, we introduced the g -circulant matrix with the generalized k -Horadam numbers and presented some properties of this matrix. By the results in Sections 2 of this paper, we have a great opportunity to obtain norm, determinant and inverse of the circulant matrices with second order number sequences. Thus, we extend some recent result in the literature. In the future studies on the circulant matrix for number sequences, we expect that the following topics will bring a new insight. For example, it would be interesting to study the g -circulant matrix for third order number sequences.

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